Journal of Mathematical Sociology, 1994, Vol. 19(1), pp. 29–52
Reprints available directly from the publisher
Photocopying permitted by license only

Printed in Malaysia

REGULAR EQUIVALENCE: GENERAL THEORY

MARTIN G. EVERETT

School of Mathematics, Statistics and Computing, University of Greenwich, Wellington Street, London SE18 6PF, UK

STEPHEN P. BORGATTI

Department of Sociology, University of South Carolina, Columbia, SC 29208, USA

June 13, 1993; revised December 12, 1993

The theory of regular equivalence has advanced over the last 15 years on a number of different fronts. Notation and terminology have developed often making it difficult to obtain a coherent view of the area as a whole. This paper attempts to provide a framework in which to develop and explore the general mathematical theory of regular equivalence and to place a number of the more important results into that framework.

1. INTRODUCTION

The purpose of this paper is to give a coherent survey of the major results known about regular equivalence. It is hoped that, in so doing, we will avoid some of the inevitable confusion which occurs when related concepts are developed in a number of separate articles using different notations and definitions. We shall only be concerned with formal definitions and mathematical results and will not address methods, techniques or algorithms for practical data analysis.

The first published article which introduces regular equivalence was by Sailer (1978). In this article the concept was called structural relatedness and was attributed by Sailer to John Boyd. The concept was developed in a working seminar at the University of California, Irvine during 1977. (This Seminar Group also contained D. R. White).

The term regular equivalence was first proposed by White and Reitz (1983). This seminal paper modified the concept of structural relatedness, making it symmetric, provided a number of important theorems and extended the ideas to networks of relations. It is this paper that gave the first insight into the rich structure provided by the regular equivalence concept.

The paper is organized so that the earlier sections explore the more general results of regular equivalence; these lead naturally into particular examples of equivalences. We next explore regular equivalences with specific properties and examine the relationships between them. These relationships can be exploited by taking certain compositions of equivalences to obtain more equivalences. For the sake of clarity all of these results are derived on a single digraph or digraph without isolates. The later sections examine the situation when these restrictions are relaxed and in the final section we look at weaker conditions of regular equivalence and examine the relationship between regular equivalence and other equivalences.

We do not provide formal proofs for any of the results, since these are contained in the original cited articles. There are a few results for which proofs are not provided in the referenced articles, these should be straightforward or are natural extensions of the published results.

2. TERMINOLOGY AND SCOPE

Let $G(V,E)$ be a finite graph with vertex set V and edge set E. Self-loops are allowed, multiple edges are also allowed but are not relevant until we introduce the concept of exact equivalence. We shall refer to directed graphs as digraphs and use the notation $D(V,E)$.

In a directed graph we define the *in-neighbourhood* of a vertex v as the set of vertices from which v receives connections, and the *out-neighbourhood* as the set of vertices which receive connections from v; these are denoted by $N_i(v)$ and $N_0(v)$ respectively, so that

$$N_i(v) = \{x : (x,v) \in E\}$$
$$N_0(v) = \{x : (v,x) \in E\}.$$

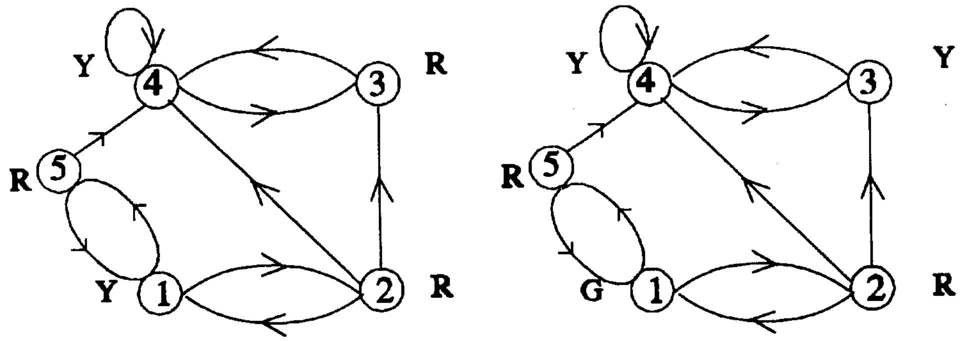
In the undirected case these two sets would be the same and we therefore use the general term *neighbourhood* of v, which will be denoted by $N(v)$. A *coloration* C of a digraph D is an assignment of colours to the vertices of D. Any coloration induces a partition of the vertices which defines an equivalence relation. Conversely, any partition or equivalence relation on the vertices induces a coloration. If S is a subset of the vertices of a digraph then the *spectrum* of S, denoted by $C(S)$, is the set of all colours assigned to the vertices of S. If S consists of just a single vertex v, then we shall write $C(v)$ and call this the *colour* of v.

Throughout this article we shall assume that definitions which are given for digraphs will also apply to graphs provided the concepts are relevant. Hence, in the above, the concepts of in and out neighbourhood was only applicable to digraphs and we therefore introduced the corresponding definition of a neighbourhood for graphs. In contrast the concept of coloration is applicable to both graphs and digraphs and so we have not provided a separate 'graph' definition.

3. REGULAR COLORATION

A coloration C of a digraph $D(V,E)$ is *regular* if and only if for all $u, v \in V$

$$C(u) = C(v) \Rightarrow C(N_i(u)) = C(N_i(v))$$
$$\text{and} \qquad C(N_0(u)) = C(N_0(v)).$$



(i) (ii)

FIGURE 1.

In the undirected case this reduces to the condition that

$$C(u) = C(v) \Rightarrow C(N(u)) = C(N(v)).$$

The equivalence relation induced by a regular coloration is called a *regular equivalence.* The undirected version is the same as the structural relatedness concept introduced in the paper by Sailer (1973); the directed version corresponds to the definition of regular equivalence given by White and Reitz (1983). The formulation in terms of coloration was proposed by Everett and Borgatti, (1991).[1]

Figure 1 represents two colorations of the same digraph. The colours of each vertex are given by capital letters, such as "R" and "Y". If we examine the two colorations in the figure, we can see that coloration (i) is not regular whereas (ii) is. In coloration (i) we see that $C(2) = C(3) = \{R\}$ but that $N_i(3) = \{2,4\}$ and $N_i(2) = \{1\}$ hence $C(N_i(2)) = \{Y\}$ whereas $C(N_i(3)) = \{R,Y\}$. This violates the condition since two vertices coloured red have different spectra for their in-neighbourhoods. In contrast (ii) has only two pairs of vertices coloured the same. If we examine 3 and 4 we see that

$$N_i(3) = \{2,4\} \quad \text{and} \quad C(N_i(3)) = \{R,Y\}$$

$$N_0(3) = \{4\} \quad \text{and} \quad C(N_0(3)) = \{Y\}$$

$$N_i(4) = \{2,3,4,5\} \quad \text{and} \quad C(N_i(4)) = \{R,Y\}$$

$$N_0(4) = \{3,4\} \quad \text{and} \quad C(N_0(4)) = \{Y\}$$

and so satisfies the definition. Note that the use of sets means that we do not insist that the connections are to an identical number of colours but simply that there is a connection to at least one colour. For example $N_i(4)$ contains 4 elements whereas $N_i(3)$ contains 2 elements. They do, however, have the same spectra.

In any graph or digraph, the coloration which assigns a different colour to every vertex is always regular. We shall now assume that all graphs and digraphs do not

[1]They used the term "colouring" where we use "coloration." The present usage brings the terminology in line with standard graph theory, in which "coloration" refers to the general concept and "colouring" refers to a specific coloration in which adjacent nodes are coloured differently.

contain isolates (i.e. vertices with no adjacencies to other vertices). This restriction is of little consequence to the theory of regular coloration but it does simplify the statements of a number of the results.

In a graph (but not necessarily a digraph) the coloration in which every vertex is coloured differently is also regular. It is clear that for any graph or digraph there may be a number of possible regular colorations. Since every regular coloration defines a partition (two vertices, being in the same colour class if they are coloured the same) then we can order the set of regular colorations by using the refinement relation on the induced partition. Let C be the set of all regular colorations of a digraph D. Let $C_1, C_2 \in C$ and suppose $C_1 \leq C_2$, where $\leq$ is the order defined above, then for any pair of vertices u and v in D such that $C_1(u) = C_1(v)$ it follows that $C_2(u) = C_2(v)$.

THEOREM 1 (Borgatti and Everett, 1989) *The class of all regular colorations form a lattice under the partial order* $\leq$.

Although the join of the lattice of regular colorations is the same as in the lattice of all colorations, the meet is formed differently, and hence the regular lattice is not a sublattice of the lattice of all colorations. Figure 2 gives a simple graph together with its lattice of regular colorations; the colorations are given in terms of the induced partitions.

For a graph, the maximum and minimum elements of the lattice are the colorations which colour each vertex the same and each vertex differently, respectively. For a digraph the minimal element is the same but the maximum may not be the one that colours all vertices the same. The following theorem tells us when this is the case. We first need some simple definitions. A *source* is a vertex whose in-neighbourhood is empty and a *sink* is a vertex whose out-neighbourhood is empty.

THEOREM 2 (Borgatti and Everett, 1989) *In a digraph D, colouring every vertex the same colour is a regular coloration if and only if D contains no sources or sinks.*

We call the colorations which colour all vertices the same, or all vertices different, *trivial colorations*.

Early work in regular equivalence focused upon the maximal element of the lattice and many users of regular equivalence use the term simply to refer to this element. We shall call this coloration the *maximal* regular coloration. The above theorem tells us that for graphs and certain digraphs this maximal element is one of the trivial colorations and, as such, is of no mathematical or substantive interest. It is natural to focus our attention on the non-trivial regular colorations. However, it is quite possible for a graph to have no non-trivial colorations. We call such graphs or digraphs *role-primitive*. The graph in Figure 3, discovered by Erik Jacobson (personal communication), is the smallest known graph with no non-trivial colorations.

If we form a digraph from the graph in Figure 3 by simply replacing an edge (u, v) by two directed edges (u, v) and (v, u), we would obtain a role primitive digraph. It is possible for a digraph to be role primitive with only one trivial regular coloration, since colouring each vertex of a digraph the same colour is not necessarily regular. Figure 4 is an example of such a digraph.

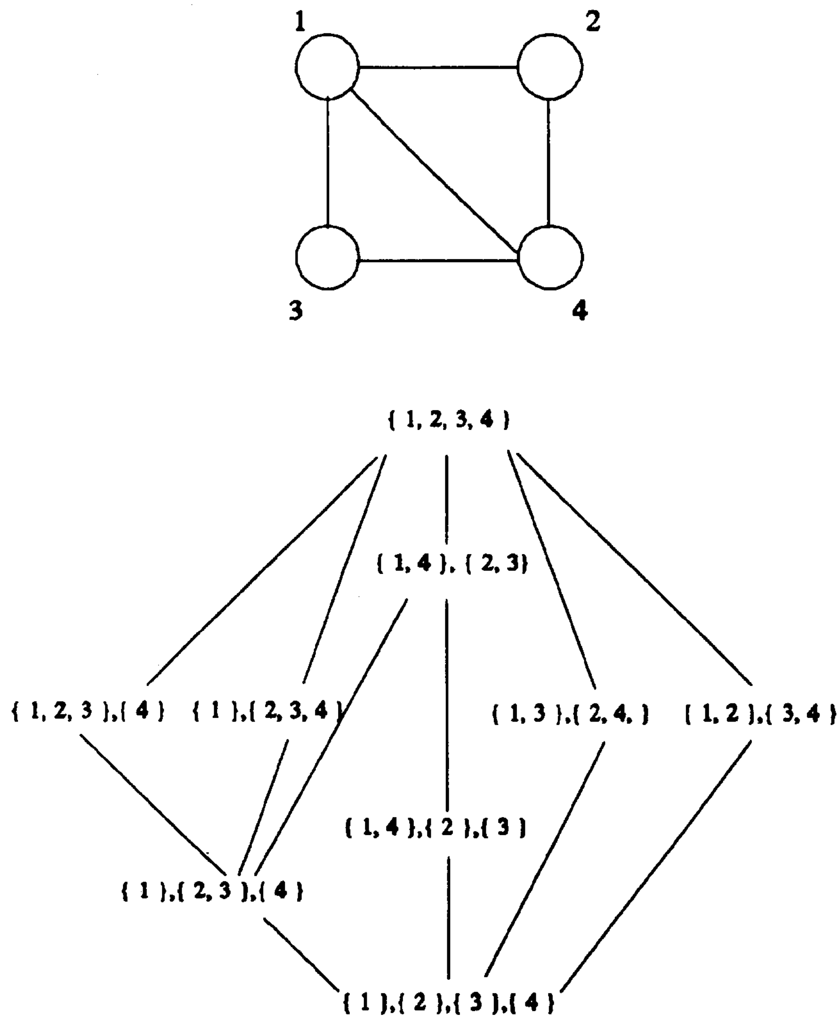


FIGURE 2.

Let $D(V,E)$ be a labelled digraph. An *automorphism* of D is a permutation π of V such that $(u,v) \in E$ if, and only if $(\pi(u),\pi(v)) \in E$. In other words an automorphism of a digraph or graph is a permutation of the vertices which preserves adjacency. For the graph in Figure 2 the permutation $\pi(1) = 4$, $\pi(2) = 3$, $\pi(3) = 2$, $\pi(4) = 1$ is an automorphism. A digraph in which the only automorphism is the identity permutation is called an *identity digraph.*

THEOREM 3 (Everett and Borgatti, 1991) *Every role primitive graph is an identity graph.*

This theorem does not extend to digraphs. Any odd directed cycle of length n is role primitive but the permutation which maps each vertex to its only out-neighbour is an automorphism. (There are in fact n automorphisms for such a digraph).

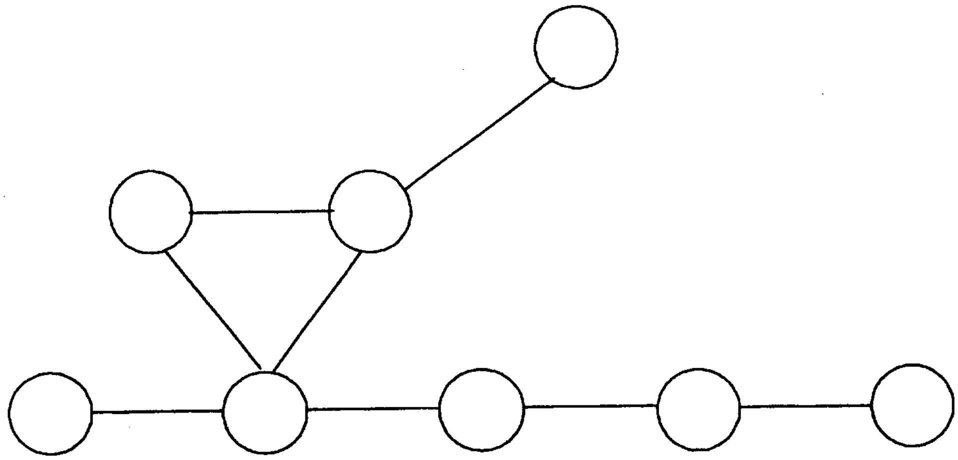


FIGURE 3.

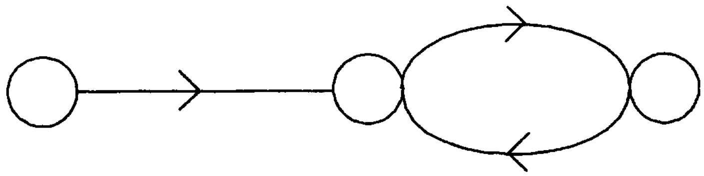


FIGURE 4.

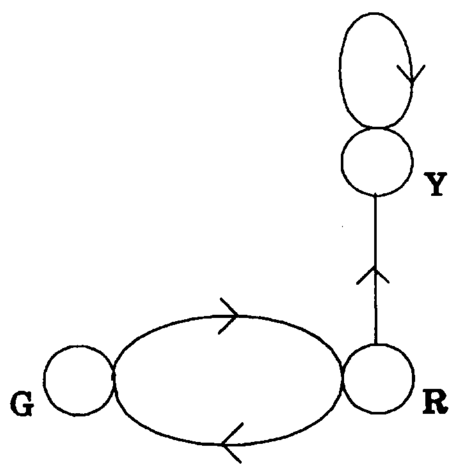


FIGURE 5.

Let $D(V,E)$ be a digraph with coloration C. The *image digraph* $D'(C(V),E')$ has the spectrum of V as its vertices; two vertices are adjacent in D' if there exists an edge between nodes of those colours in D. Figure 5 is the image digraph of Figure 1(ii).

Note that the imag
to the vertices in Figu
to form a new colorat
original digraph. Supp
the image $D'(C(V),E$
assigns the colour $C'($

THEOREM 4 (Borgatti
digraph D and C' is
coloration of D.

Any digraph $D(V,E$
use E to generate a se
call this semigroup the

THEOREM 5 (White an
a digraph D. Let S and
S' is a homomorphic i

This theorem provi
social roles, namely th
Any digraph can be
all the ties. We shall c
is normal practice to r
does not create a multi
(However, for the pur
convention is obeyed c

THEOREM 6 (Everett a
regular coloration of th

The graph in Figure

4. CHARACTERIS

We have used graph co
regular equivalence. T

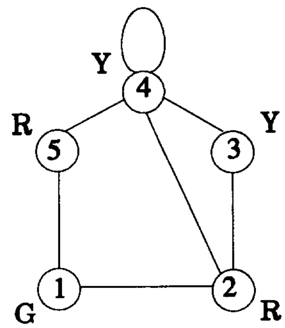


FIGURE 6.

Note that the image digraph does not have a coloration. The colours attached to the vertices in Figure 5 are labels. It follows that we can now colour the image to form a new coloration. The coloration of the image induces a coloration of the original digraph. Suppose that C is a coloration of $D(V,E)$ and C' is a coloration of the image $D'(C(V),E)$. Then for every vertex $v \in V$ the *induced coloration* $C' \circ C$ assigns the colour $C'(C(v))$.

THEOREM 4 (Borgatti, Boyd and Everett, 1989) *If C is a regular coloration of a digraph D and C' is a regular coloration of the image D' then $C' \circ C$ is a regular coloration of D.*

Any digraph $D(V,E)$ is simply a binary relation E on the vertex set V. We can use E to generate a semigroup S under the operation of relational composition. We call this semigroup the *semigroup associated* with the digraph D.

THEOREM 5 (White and Reitz, 1983) *Let D' be the image of a regular coloration of a digraph D. Let S and S' be the associated semigroups of D and D' respectively then S' is a homomorphic image of S.*

This theorem provides a link between the two main approaches to modelling social roles, namely the use of semigroups and graph theory.

Any digraph can be converted to a graph by simply ignoring the directionality of all the ties. We shall call the graph formed by this process the *underlying graph.* It is normal practice to replace reciprocal edges by just one edge so that the process does not create a multigraph from a digraph which does not possess multiple edges. (However, for the purposes of regular coloration, it does not matter whether this convention is obeyed or not).

THEOREM 6 (Everett and Borgatti, 1993) *Any regular coloration of a digraph is a regular coloration of the underlying graph.*

The graph in Figure 6 shows this result for the digraph in Figure 1(ii).

4. CHARACTERISATIONS

We have used graph coloration as our principal vehicle for expressing results about regular equivalence. There are a number of alternate definitions which could be used. Any practitioner who wishes to use regular equivalence—whether to further develop the theory, construct algorithms or analyze data—would benefit from being aware of the alternatives. They provide flexibility of interpretation and application and can often be used to relax the strict conditions of regular equivalence to allow for the development of practical algorithms for the analysis of real data.

The first published definition of regular coloration in the form it is used today was in terms of an equivalence on a digraph.

THEOREM 7 (White and Reitz, 1983) *Let $D(V,E)$ be a digraph and $\equiv$ an equivalence relation on V. If $\equiv$ is such that for all $u,v,w \in V$, $u \equiv v$ implies both*

(i) *$(u,w) \in E$ implies there exists $x \in V$ such that $(v,x) \in E$ and $x \equiv w$,*
(ii) *$(w,u) \in E$ implies there exists $x \in V$ such that $(x,u) \in E$ and $x \equiv w$,*

then $\equiv$ induces a regular coloration in D. Conversely, if $\equiv$ is an equivalence relation induced by a regular coloration then $\equiv$ satisfies the conditions given above.

The two conditions correspond to the in and out neighbourhoods of the coloration formulation.

If we examine the digraph Figure 1(ii) we see that $3 \equiv 4$; since $(2,3) \in E$ this implies there exists an $x \in V$ such that $(x,4) \in E$ and $2 \equiv x$, $x = 5$ satisfies this condition (note that $x = 2$ also satisfies the condition).

Theorem 7 simply replaces the coloration by the induced equivalence classes. The equivalence classes are equally specified by the image digraph, this leads to the next characterisation.

THEOREM 8 (Borgatti, 1989; Everett and Borgatti, 1993) *Let $D(V,E)$ be a digraph with coloration C, then C is a regular coloration if and only if for every $v \in V$, $C(N_0(v)) = N_0(C(v))$, and $C(N_i(v)) = N_i(C(v))$. Note that the right hand side refers to the neighbourhood of the image digraph under C.*

This theorem is illustrated by comparing Figure 1(ii) with Figure 5. For example we see from Figure 5 that $N_0(R) = \{G,Y\}$ and that the two vertices coloured R in Figure 1(ii), (7 and 5) both have out-neighbourhoods with $\{G,Y\}$ as their spectrum.

Our next characterisation moves away from the graph theoretic representation of the data and looks at a matrix formulation. The characterisation relies on the concept of a block which was first introduced as a characterisation of a more specific form of regular equivalence, namely structural equivalence. (Breiger et al., 1975, White et al., 1976). A block of a matrix is the matrix which remains when rows and/or columns are deleted. A blocking is the set of blocks of a matrix corresponding to a partitioning of the rows and or columns. When the matrix is an adjacency matrix then the partitioning will correspond to a partition of the vertices so that the rows and columns have the same partition. We shall assume, without loss of generality, that the rows and columns of the matrix have been permutated so that the equivalence classes of the partition are grouped together. We shall call a partition *regular* if it induces a regular equivalence.

THEOREM 9 *A part*
blocks of the correspo
contain a one in every

Theorem 9 is illust
It is easy to see that t
or have a one in ever

The blocking formula
independently by Borg
multiway and multimo
Our final character
alence matrix, *E*, has
equivalent; and zero o

THEOREM 10 (Boyd,
regular on a digraph w

under Boolean matrix

We can again illustra
the equivalence relatio

$A =$

THEOREM 9 *A partitioning of the vertices of digraph is regular if and only if the blocks of the corresponding blocking of the adjacency matrix consist either of zeros, or contain a one in every row and every column.*

Theorem 9 is illustrated on the adjacency matrix A corresponding to Figure 1(ii). It is easy to see that the 9 blocks each have the property that they are either all zero or have a one in every row and every column.

$$\begin{array}{c|c|cc|cc|}
 & 1 & 2 & 5 & 3 & 4 \\ \cline{2-6}
1 & 0 & 1 & 1 & 0 & 0 \\ \cline{2-6}
2 & 1 & 0 & 0 & 1 & 1 \\
5 & 1 & 0 & 0 & 0 & 1 \\ \cline{2-6}
3 & 0 & 0 & 0 & 0 & 1 \\
4 & 0 & 0 & 0 & 1 & 1 \\ \cline{2-6}
\end{array}$$

The blocking formulation was derived as a characterisation of regular equivalence independently by Borgatti (1989) and Batagelj et al. (1992). An extension to valued, multiway and multimode matrices is found in Borgatti and Everett (1992c).

Our final characterisation also uses the adjacency matrix. We define an *equivalence matrix*, E, has a matrix that has one row i and column j if i and j are equivalent; and zero otherwise.

THEOREM 10 (Boyd, 1991) *An equivalence relation, with equivalence matrix E, is regular on a digraph with adjacency matrix A if and only if*

$$EA = AE$$

under Boolean matrix multiplication.

We can again illustrate this theorem on the digraph in Figure 1(ii). We have, using the equivalence relation induced by the coloration, the following

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$AE = EA = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

5. STRUCTURAL EQUIVALENCE

The concept of structural equivalence was introduced into the network literature by Lorrain and White (1971) using the notation and terminology of category theory. It was part of a two stage recursive process to uncover the role structure of a network by first identifying (i.e. equating) equivalent vertices and then reducing the semigroups associated with the network. Modern usage has only been concerned with the first stage of that process as applied directly to raw data. We shall give an equivalent graph theoretic definition of structural equivalence, which for reasons which will soon become clear we call strong structural equivalence. Two vertices, u, v of a digraph are *strongly structurally equivalent* if and only if $N_i(u) = N_i(v)$ and $N_0(u) = N_0(v)$. Hence strongly structurally equivalent vertices have the same neighbourhoods. Note that strong structural equivalence is an equivalence relation and so we can partition the vertices into equivalence classes. Over time, researchers have come to see these classes as a definition of structural equivalence, with the consequence that any digraph is seen as having exactly one structurally equivalent partition.

In contrast, we focus on the fact that structural equivalence is a binary relation, from which we define a sub-lattice of structural coloration, as given by the following three theorems.

THEOREM 11 *In a digraph D any coloration in which vertices that are coloured the same are strongly structurally equivalent, is a regular coloration.*

We call such a coloration a *strong structural coloration.*

In Figure 7 the vertices 1, 2 and 3 are strongly structurally equivalent and so are 4 and 5. It follows from the theorem that the coloration given by the colour classes $\{1,3\}$, $\{2\}$, $\{4,5\}$ and $\{1,2,3\}$, $\{4,5\}$ are both regular. We shall call the coloration which corresponds to the equivalence classes of strong structural equivalence the *maximal strong structural equivalence.* In Figure 7, this is the coloration with colour classes $\{1,2,3\}$, $\{4,5\}$.

One of the difficulties of the concept of using equal neighbourhoods occurs in digraphs without self-loops. If we remove the self-loops of the digraph on Figure 7 then 4 and 5 are no longer strongly structurally equivalent since their neighbourhoods are now different. The concept of weak structural equivalence overcomes this. Two vertices u and v of a digraph D are *weakly structurally equivalent* if, and only if, the permutation $\pi = (u, v)$ is an automorphism. So that π is the permutation which exchanges the labels of u and v.

THEOREM 12 *In a digraph D, any coloration in which vertices that are coloured the same are weakly structurally equivalent, is a regular coloration.*

We call such a coloration a *weak structural coloration.*

Any pair of vertices which are strongly structurally equivalent are weakly structurally equivalent but the converse is false. In the graph in Figure 8 the vertices 4 and 5 are weakly structurally equivalent but not strongly structurally equivalent.

THEOREM 13 *The class of all weak (strong) structural colorations form a sub-lattice of the lattice of regular colorations.*

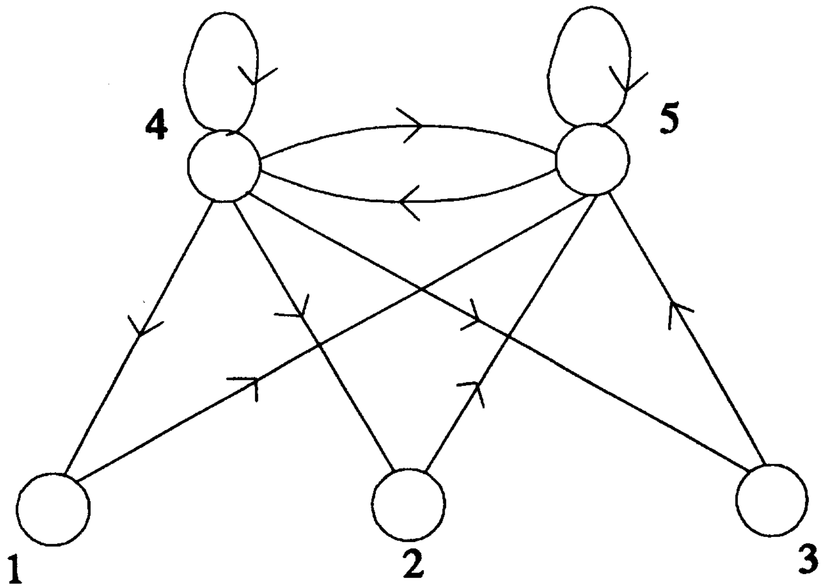


FIGURE 7.

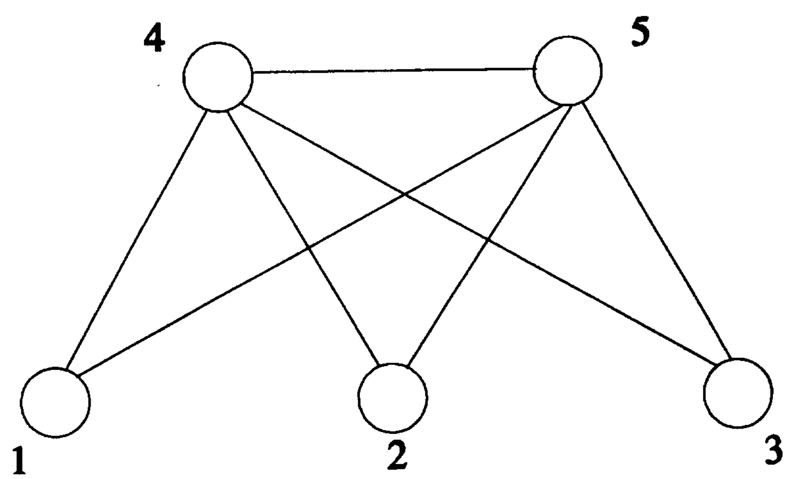


FIGURE 8.

Considered as lattices in their own right, both are mathematically uninteresting since any refinement of the maximal structural coloration is a structural coloration.

THEOREM 14 (Borgatti, Boyd and Everett, 1989) *If D' is the image digraph of a maximal weak (strong) coloration then D' contains no weakly (strongly) structurally equivalent vertices.*

It is possible for the image digraph of a maximal strong structural coloration to have vertices which are weakly structurally equivalent. Figure 9 is the image graph of the maximal strong coloration of Figure 8, this coloration has $C(4) = R$,

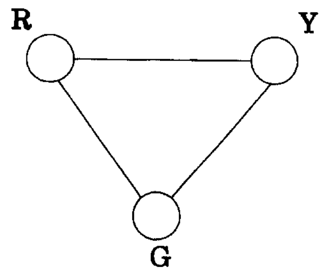


FIGURE 9.

$C(5) = Y$, $C(1) = C(2) = C(3) = G$. All the vertices in Figure 9 are weakly structurally equivalent.

The following result was communicated to us by John Boyd.

THEOREM 15 *An equivalence relation, with equivalence matrix E, is a weak structural equivalence on a digraph with adjacency matrix A if and only if*

$$EA = A \quad \text{and} \quad AE = A$$

under Boolean matrix multiplication.

6. AUTOMORPHIC EQUIVALENCE

Automorphic equivalence was first proposed by Chris Winship in a paper written in 1974 but not published until 1988. It was introduced in a joint paper by Winship and Mandel (1983) but the idea was not fully explored and the mathematical definitions were relegated to footnotes. The concept was independently suggested by Everett (1985) who also demonstrated that it was a regular equivalence. Two vertices u and v of a digraph D are *automorphically equivalent* if and only if there exists an automorphism π such that $\pi(u) = v$. Automorphic equivalence is a natural generalization of weak structural equivalence. The underlying principle involved is based upon the observation that two vertices in an unlabelled digraph which are indistinguishable are structurally alike.

It is not the case that any coloration in which vertices that are coloured the same are automorphically equivalent forms a regular coloration. This can be seen by examining Figure 10. The vertices 1 and 6 are automorphically equivalent, however if we colour them the same and all other vertices a different colour, we do not have a regular coloration. The reason is that any automorphism which maps 1 to 6 must also map 2 to 4 and we need to use this to obtain a regular coloration.

The collection of all automorphism forms a group under the operation of composition called the *automorphism group* and denoted by Aut(D). The graph in Figure 10 has four automorphisms:

1. the identity,
2. (3 5),
3. (1 6)(2 4),
4. (1 6)(2 4)(3 5).

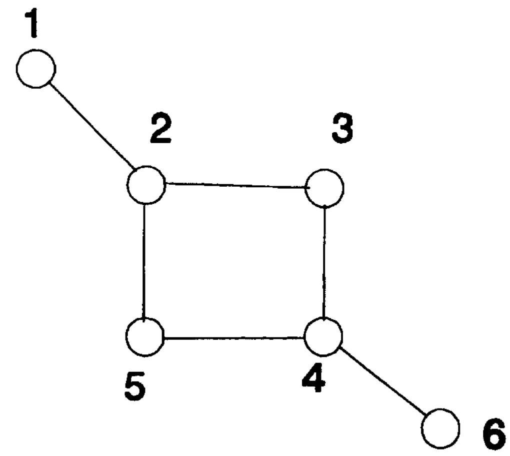


FIGURE 10.

The two vertices u and v belong to the same orbit of a permutation group P if and only if $\pi(u) = v$ for some $\pi \in P$. The orbits of the permutation group are the equivalence classes of the equivalence relation of automorphic equivalence, where the automorphisms are only taken from P.

THEOREM 16 (Everett, 1985) *Let $D(V,E)$ be a digraph and H a subgroup of the automorphism group* Aut(D). *Colouring each orbit of H a different colour gives a regular coloration of D.*

We call a coloration formed in the manner described in Theorem 16 an *orbit* coloration.

The subgroup could be the whole automorphism group and these are often called the orbits of the digraph. For the graph in Figure 10 this would produce the regular coloration with colour classes $\{1,6\}$, $\{2,4\}$ and $\{3,5\}$. The automorphism group has two non-trivial subgroups which produce regular colorations with colour classes $\{1,6\}$, $\{2,4\}$, $\{3\}$, $\{5\}$ and $\{3,5\}$, $\{1\}$, $\{2\}$, $\{4\}$, $\{6\}$.

This example illustrates a useful source of subgroups, namely those generated by an individual permutation. The cycles in the cyclic representation of a permutation are the orbits of the subgroup generated by that permutation.

The set of automorphisms which leaves a vertex unchanged also forms a subgroup called the *stabilizer* of v. Winship and Mandel (1983) claim that the coloration corresponding to the orbits of the stabilizer of a vertex v gives a model of the role system from v's perspective. The first of the non-trivial subgroups gives the stabilizers of 3 and 5 whereas the second gives the stabilizers of 1, 2, 4 and 6.

Our next result is similar to the general result for regular coloration expressed in Theorem 10.

A permutation π which acts on the set of vertices $v_1, \ldots, v_n$ can be represented by a permutation matrix P, where $P_{ij} = 1$ if $\pi(v_i) = v_j$ and $P_{ij} = 0$, otherwise. The following result is standard.

THEOREM 17 *Let A be an adjacency matrix for a digraph and P a permutation matrix. Then P is an automorphism if and only if $AP = PA$.*

7. PERFECT EQUIVALENCE

Borgatti and Everett (1992b) introduced the concept of ecological coloration to help analyze power in experimental exchange networks. A coloration is ecological if

$$C(N_i(u)) = C(N_i(v)) \quad \text{and} \quad C(N_0(u)) = C(N_0(v)) \Rightarrow C(u) = C(v).$$

This is precisely the converse of the regular coloration condition so that a vertex's colour is determined by the spectrum of its neighbourhood. The coloration of the digraph in Figure 1(i) has no neighbourhoods with the same spectrum and so satisfies the condition vacuously. The regular coloration in Figure 1(ii) is also an ecological coloration but in this case there are two pairs of vertices (3 and 4, and 2 and 5) which have neighbourhoods with the same spectrum.

The automorphic coloration for the graph in Figure 10 (i.e. colour classes $\{1,6\}$, $\{2,4\}$ and $\{3,5\}$) is a regular coloration which is not ecological. In particular, the spectra of the neighbourhoods of 1 and 5 are identical yet the vertices themselves are not coloured the same, which violates the ecological condition.

As in the regular coloration case, we can again order the colorations by the refinement relation.

THEOREM 18 (Borgatti and Everett, 1993) *The class of all ecological colorations forms a lattice under the partial order $\leq$.*

The maximum element of this lattice is the coloration in which every vertex is coloured the same. The minimum element is the maximal strong structural coloration. We call a coloration which is both regular and ecological a *perfect* coloration. The following characterization theorem is useful in determining whether a regular coloration is perfect.

THEOREM 19 (Borgatti and Everett, 1993) *A regular coloration is perfect if and only if the image digraph contains no strongly structurally equivalent vertices.*

It follows that the maximal strong structural coloration and the maximum regular coloration are both perfect.

Unlike regular colorations (see Theorem 4), compositions of ecological colorations are not necessarily ecological. However, the combination of ecological and regular colorations do have this property, which is formally stated in the next theorem.

THEOREM 20 (Borgatti and Everett, 1993) *If C is a perfect coloration of a digraph D and C' is a perfect coloration of the image D', then $C' \circ C$ is a perfect coloration of D.*

Since both ecological and regular colorations form lattices under the refinement ordering, it is reasonable to assume that perfect colorations also form a lattice.

THEOREM 21 (Borgatti and Everett, 1993) *The class of all perfect colorations forms a lattice under the partial order $\leq$.*

The maximum element of the lattice is the maximal regular coloration and the minimum element is the maximal strong structural coloration. It is interesting to note that the lattice of perfect colorations is not a sub-lattice of either the regular or the ecological coloration lattices.

8. EXACT COLORATIONS

In the context of social network analysis, exact colorations were discussed by Borgatti and Everett (1992b), but have not been formally defined. In the graph theory literature, however, exact colorations have been known for some time. A coloration is *exact* if whenever two vertices are coloured the same, their neighbourhoods contain not only the same sets of colours, but the same number of vertices of each colour present. Hence, in an exact coloration, if a blue vertex is connected to three reds and a yellow, all other blue vertices are also connected to exactly three reds and a yellow, and nothing else.

Let $D(V,E)$ be a digraph with coloration C. If $k \in C(V)$, we define the *k-coloured out-neighbourhood* of a vertex v, denoted by $N_0(v,k)$, as the set of vertices adjacent to v with colour k. We define $N_i(v,k)$ similarly so that

$$N_i(v,k) = \{x : (x,v) \in E \text{ and } C(x) = k\} \qquad \text{and}$$

$$N_0(v,k) = \{x : (v,x) \in E \text{ and } C(x) = k\}.$$

The *k-coloured out-degree* of a vertex v, denoted by $\rho_0(v,k)$, is the number of vertices adjacent to v coloured k. We define the *k-coloured in-degree* similarly, so that

$$\rho_i(v,k) = |N_i(v,k)| \qquad \text{and} \qquad \rho_0(v,k) = |N_0(v,k)|.$$

We define k-coloured neighbourhoods and k-coloured degrees for undirected graphs in the obvious way.

A coloration C of a digraph $D(V,E)$ is exact if and only if for all $u,v \in V$ and every $k \in C(V)$

$$C(u) = C(v) \Rightarrow \rho_i(u,k) = \rho_i(v,k) \qquad \text{and} \qquad \rho_0(u,k) = \rho_0(v,k).$$

In the undirected case this reduces to the condition

$$C(u) = C(v) \Rightarrow \rho(u,k) = \rho(v,k).$$

It is easy to see that any exact coloration must be regular but the converse is false. The regular coloration of Figure 1(ii) is not exact since, for example, $\rho_i(4,R) = 2$ and $\rho_i(3,R) = 1$, but 3 and 4 are both coloured the same. Weak and strong structural colorations are exact as is the orbit coloration described in Theorem 16. The coloration given in Figure 11 is an exact coloration which is not derived from orbits or structural equivalence. Note that the concept of exact colorations can be applied to multigraphs.

As with other colorations we have considered, the collection of exact colorations form a lattice under the refinement ordering.

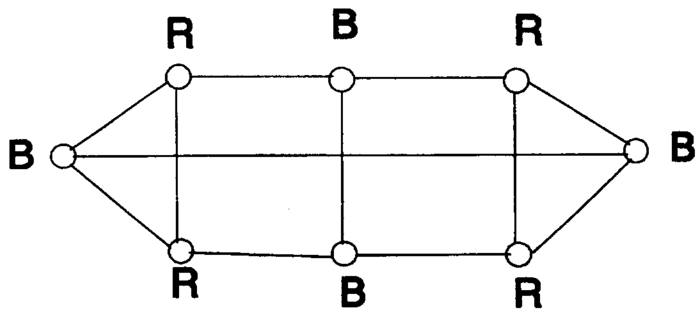


FIGURE 11.

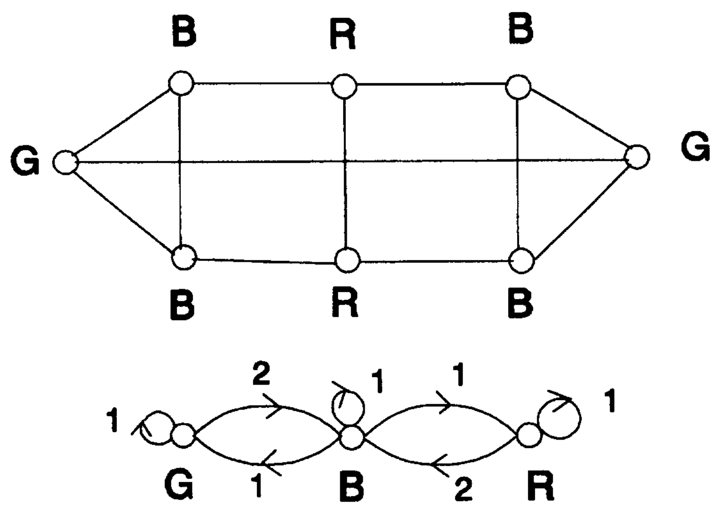


FIGURE 12.

THEOREM 22 (Hirschman and Everett, n.d.) *The class of all exact colorations forms a lattice under the partial order* $\leq$.

We can introduce a more sophisticated image digraph for exact colorations called the enhanced image digraph. Let $D(V,E)$ be a digraph with exact coloration C. The *enhanced image digraph* $D^*(C(V),E^*)$ has the spectrum of V as its vertices. Two vertices A and B have an edge from A to B of multiplicity R if a vertex coloured A is connected to R vertices of colour B in D. Figure 12 gives an exact coloration together with its enhanced image digraph.

We note that an undirected graph gives an enhanced image digraph which is both directed and has multiple edges. Composition of exact colorations still produces an exact coloration.

THEOREM 23 *If C is an exact coloration of a digraph D, and C^* is an exact coloration of the enhanced image D^*, then $C^* \circ C$ is an exact coloration of D.*

The eigenvalues and characteristic polynomial of a digraph are simply the eigenvalues and characteristic polynomial of its adjacency matrix.

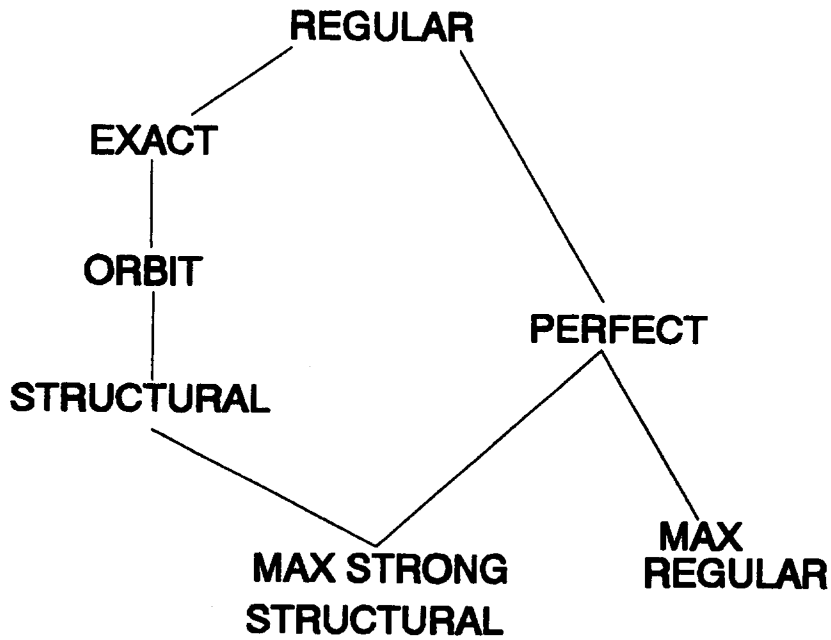


FIGURE 13.

There is an important relationship between the characteristic polynomial of a digraph with an exact coloration and the characteristic polynomial of the enhanced image digraph.

THEOREM 24 (Sachs, 1966) *The characteristic polynomial of the enhanced image digraph of an exact coloration divides the characteristic polynomial of the original digraph.*

It is for this reason that exact colorations are called *divisors* in the standard graph theory literature.

The relationships between the various types of regular coloration are summarized by the diagram given in Figure 13. Each type of coloration is a member of the class specified above it.

Note that the maximum strong structural coloration is a regular perfect exact coloration.

8. NETWORKS AND HYPERGRAPHS

A network is a collection of digraphs with a common vertex set. We shall denote a network with vertex set V and edge set $\mathsf{R} = \{R_i\}_{i \in I}$ by $D(V, \mathsf{R})$. White and Reitz (1983) developed an extensive theory of regular equivalence for networks in their original paper. This was formulated in terms of coloration by Everett and Borgatti (1993).

The general method for dealing with networks is to colour the vertex set V and consider the coloration of each digraph separately. In this respect all the results given in the previous sections are applicable to networks provided they are interpreted in terms of each individual digraph. We do not, however, restrict our individual digraphs simply to $D(V,R_i)$ $i \in I$ but create related networks which provide network generalizations of regular coloration.

A *regular network coloration* of a network $D(V,\mathbf{R})$ is a coloration in which $D(V,R_i)$ is regular for each i. It follows that all the theorems discussed so far are applicable to each digraph $D(V,R_i)$. The *image network* $D'(C(V),\mathbf{R}')$ of a coloration C of a network $D(V,\mathbf{R})$ consists of the image digraph $D'(C(V),R_i')$ for every $i \in I$. It is possible that digraphs in the image network are isomorphic, in this case we can form a new network by unioning relations in the original network which have isomorphic images. We call this new network the *colour reduced network* and denote it by $CR(D)$. Hence if C is a coloration of $D(V,\mathbf{R})$ in which

$$D'(C(V),R_j') \cong D'(C(V),R_k')$$

then in $CR(D)$ we replace $D(V,R_j)$ and $D(V,R_k)$ by $D(V,R_j \cup R_k)$.

We call a regular network coloration of $CR(D)$ a *weak regular network coloration* of D. Obviously any regular network coloration is a weak regular network coloration, but the converse is false. Figure 14 illustrates the process; the three-relation network has a coloration in which the image network has two isomorphic graphs R_1' and R_2'. We form a colour reduced network by unioning R_1 and R_2, this now forms a regular coloration, hence the original coloration of the network is a weak regular network coloration.

It is sometimes considered desirable to have the same sets of relations connecting equivalently coloured vertices. This can be achieved by again creating a new network which encapsulates the collections of relations between vertices. Given a network $D(V,\mathbf{R})$ and $u,v \in V$, then we define the *bundle* of relations from u to v as

$$B_{uv} = \{R_i : uR_iv\} \qquad R_i \in \mathbf{R}.$$

Let $\{M_i\}_{i \in I}$ be the set of all non-empty bundles. For each M_i, define a digraph with vertex V and edge set $\tilde{M}_i$ where $(u,v) \in \tilde{M}_i$ if and only if $B_{uv} = M_i$. We call the network $D(V,\tilde{M}_i)$ the *multiplex network* of D and denote it by $MPX(D)$.

If D is a network and the coloration C of the vertices forms a regular network coloration on $MPX(D)$, then we call C a *multiplex regular coloration.*

THEOREM 25 (White and Reitz, 1983) *Every multiplex regular coloration is a regular network coloration.*

The converse of this theorem is false.

A *hypergraph* consists of a set V together with a collection A of non-empty subsets of V with the property that every member of V is contained in one of the sets in A. We denote a hypergraph by $H(V,A)$, and call the elements of V vertices and the members of A edges. Note that every graph (but not digraph) can be considered a hypergraph. The *incidence matrix* of a hypergraph is a binary matrix in which $E(i,j) = 1$ if vertex i is a member of edge j and $E(i,j) = 0$ otherwise.

Given a coloration of a digraph (or hypergraph) then the *induced edge coloration* assigns to an edge the colours of the vertices incident to it.

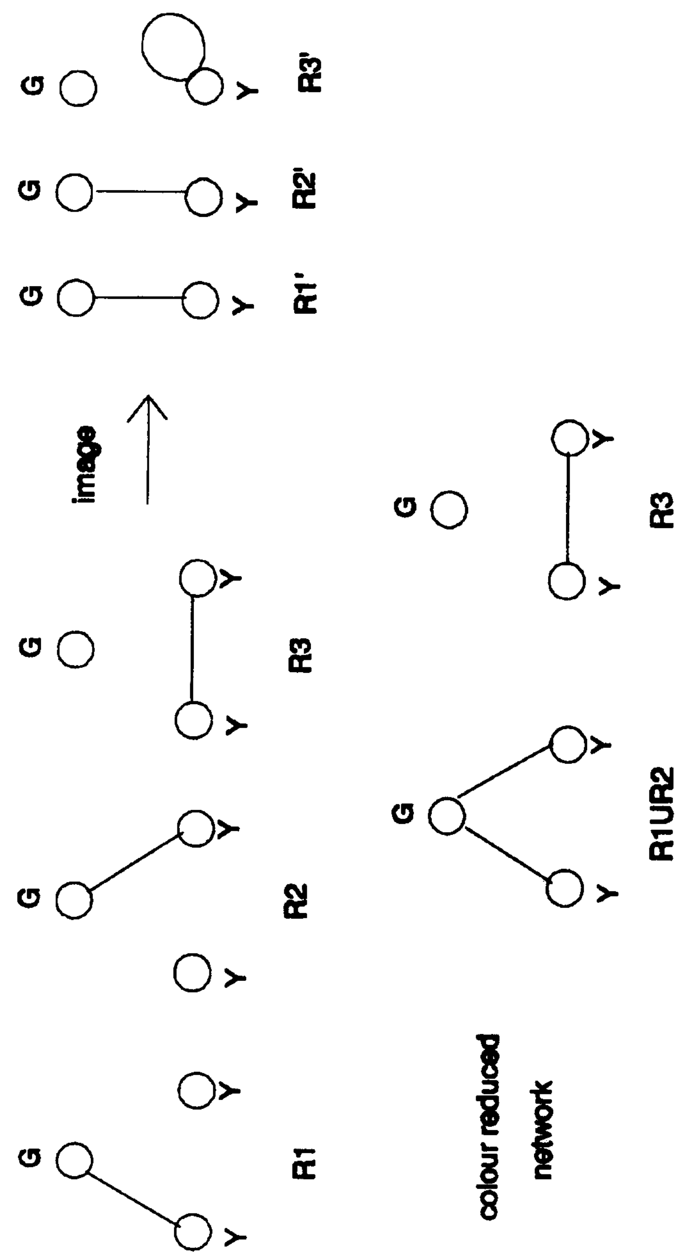


FIGURE 14.

Let E be an incident matrix for a digraph (or hypergraph) with coloration C. The *induced blocking* of E is the blocking in which the rows are partitioned into the colour classes and the columns are partitioned into the induced edge colour classes.

THEOREM 26 (Everett and Borgatti, 1993) *Let G be a digraph with incident matrix E and coloration C, if C is a regular coloration of G then the induced blocking of the incidence matrix has blocks which are either all zero or have a one in every row and every column. Conversely, any blocking of the incidence matrix of a digraph G in which every block is either all zeros or has a one in every row and every column induces a regular coloration of the vertices of G.*

We can illustrate the above on the regular coloration given in Figure 6. The regular coloration gives the partition $\{1\},\{3,4\}\{2,5\}$ which incudes the edge colour partition $\{(3,4),(4,4)\},\{(2,3),(2,4),(4,5)\},\{(1,5),(1,2)\}$. The blocked incidence matrix is therefore

	(34)	(44)	(23)	(24)	(45)	(15)	(12)
1	0	0	0	0	0	1	1
3	1	0	1	0	0	0	0
4	1	1	0	1	1	0	0
2	0	0	1	1	0	0	0
5	0	0	0	0	1	1	0

Note the 'more general' nature of the converse. We have the freedom to partition the rows and columns independently and *do not* require a partition of the vertices to induce the edge partition.

We can use the above result to provide a generalization of regular equivalence to hypergraphs. In a hypergraph a partition of the edges and vertices is called a *regular hypergraph partition* if each block of the incidence matrix is either zero or contains a one in every row and every column. The previous theorem demonstrates that when the hypergraph is a graph, the two concepts are identical.

The application of regular equivalence to incidence matrices is an instance of a more general application of regular equivalence to 2-way, 2-mode matrices. A *way* is a dimension of a matrix, such as its rows or columns. A *mode* is a set of entities referenced by a way, such as a set of persons or a set of organizations. The adjacency matrix of a digraph is a 2-way, 1-mode matrix, since both rows and columns index the same set of entities. The incidence matrix of a digraph is a 2-way, 2-mode matrix since the rows and the columns index different sets of entities. Borgatti and Everett (1992c) define 2-mode regular colorations as pairs of colorations (one for each mode) which induce a blocking of the matrix such that each matrix block is either all zeros or contains a one in each row and column (cf. Theorem 9). They then generalize this condition to the case of non-binary matrices (for every block, each row and column contains the same set of distinct values). Further extensions are possible to multiway, multimode matrices (such as customer by product by vendor matrices). These are explored in detail in Borgatti and Everett (1992c).

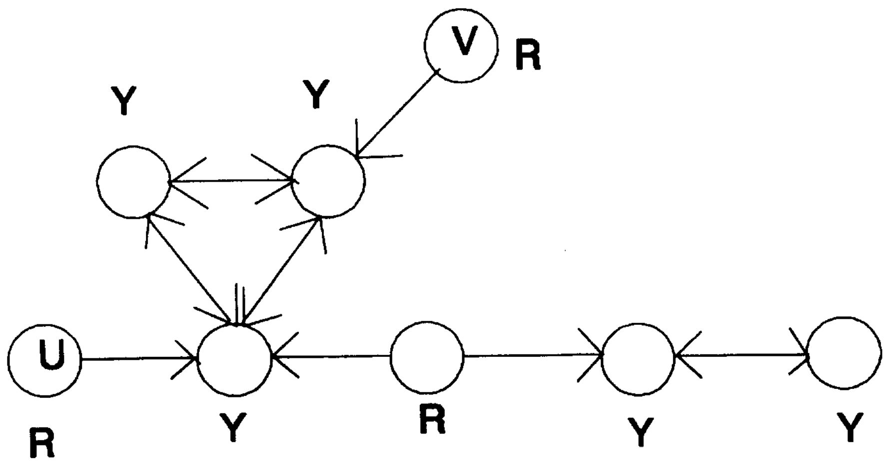


FIGURE 15.

10. WEAKER REGULAR CONDITIONS

In this section we briefly examine some of the ways in which the regular conditions have been weakened.

For directed graphs one relaxation would be to simply insist that the condition only holds in either the in or out neighbourhoods. A coloration of a digraph is called *in-regular* if and only if for all $u, v \in V$

$$C(u) = C(v) \Rightarrow C(N_i(u)) = C(N_i(v))$$

and *out-regular* if

$$C(u) = C(v) \Rightarrow C(N_0(u)) = C(N_0(v)).$$

The role primitive digraph in Figure 15 has an out-regular coloration which is not in-regular. The out-neighbourhood of red vertices are yellow and the out-neighbourhood of yellow vertices are yellow. There are however different in-neighbourhoods, for the yellow vertices all but one have red and yellow in-neighbourhoods but one yellow vertex has an in-neighbourhood which is just yellow.

These conditions were first proposed by Pattison (1982) who attributes them to J. Q. Johnson. The proofs of nearly all the results of the previous section rely on treating each condition separately and therefore nearly all the theorems remain true. Of course, each requires a slight modification to incorporate the weaker condition. The exceptions are Theorems 3, 6, 9, 14, 15, 16, 23 and 25; some of these are exceptions because the original concept is not split into the two separate in and out conditions, e.g. automorphic or hypergraph regular coloration. The exception is Theorem 23 which holds with the weaker hypothesis of an exact out-coloration. These conditions have been generalized to give a broader range of weaker conditions called the Kim and Roush condition G_i and the Kim and Roush condition G_{im}. Details are contained in Kim and Roush (1984).

An alternative weakening would be not to insist that the condition holds on the whole digraph. We can use a similar trick as for networks and construct a new

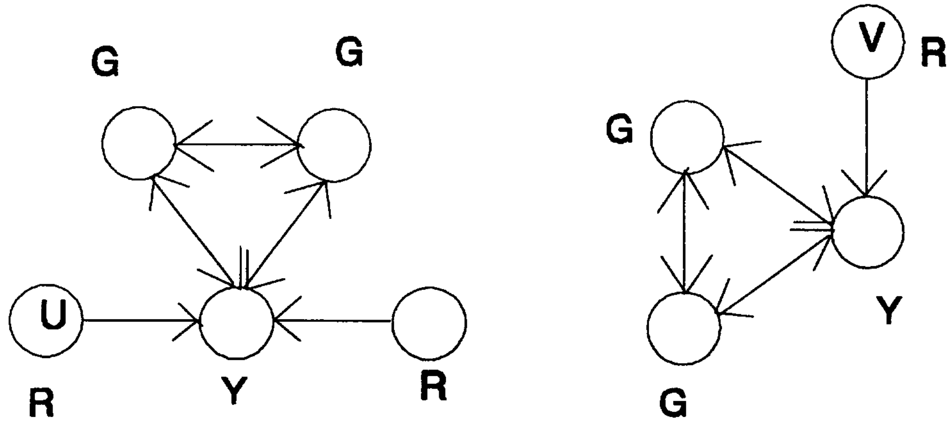


FIGURE 16.

digraph on which to apply the conditions. The *disjoint union* $A\coprod B$ of two sets A and B is defined by

$$A\coprod B = A\times\{1\}\cup B\times\{2\}.$$

The disjoint union of two digraphs is the disjoint union of their respective vertex sets and edge sets. We denote the induced subgraph of a set of vertices S by $\langle S\rangle$.

In a digraph we define $N(u) = N_i(u)\cup N_0(u)$.

Two vertices u, v in a digraph are *k-locally regularly equivalent* if there exists a regular coloration C of the digraph $\langle N^k(u)\rangle\coprod\langle N^k(v)\rangle$ in which $C(u) = C(v)$. Any pair of vertices which are regularly equivalent will be k-locally regularly equivalent for any value of k. We can again extend the definition to obtain local versions of automorphic, ecological, perfect and exact coloration. Note that we cannot use this technique for structural equivalence since the constructed digraph is not connected. It is clearly possible for vertices to be k-locally regularly equivalent and not be regularly equivalent. The relation of being k-locally regularly equivalent is not transitive, it follows that if a pair of vertices are k-locally regularly equivalent for every value of k then they still may not be regularly equivalent. If we examine the two vertices labelled u and v in Figure 15, then Figure 16 gives a regular coloration of $\langle N^2(u)\rangle\coprod\langle N^2(v)\rangle$. This shows that u and v are 2-regularly equivalent. Note that some vertices appear in each component and may be coloured differently. Further details on local regular coloration are contained in Everett, Boyd and Borgatti (1990).

We lastly mention some positional methods which are not a direct relaxation of regular coloration but can be viewed in terms of the regular coloration techniques. Cook et al. (1983) state that two vertices u and v of a digraph D occupy the same position if and only if $D-u$ is isomorphic to $D-v$. Unfortunately, they do not give any clear reasons or motivation for their definition. On reading their work it seems likely that they were striving for a definition of automorphic equivalence. It is true that if u and v are automorphically equivalent, then $D-\{u\} = D-\{v\}$, but the converse is false. This is shown by the graph G in Figure 17 in which u and v are such that $G-u\simeq G-v$, but u and v are not automorphically equivalent. It should

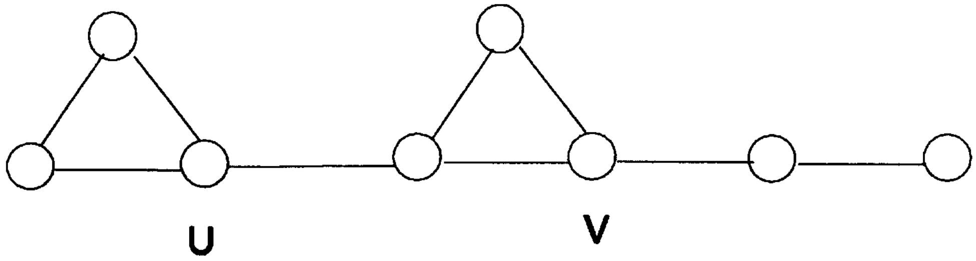


FIGURE 17.

be noted that in Cook et al. every example has automorphically equivalent vertices identified as occupying the same position. We note, however, that the Cook et al. definition is a relaxation of automorphic equivalence.

Burt (1990) provides information on a technique proposed in a paper by Hummell and Sodeur (1987) which uses a triad census technique to define role equivalence. The technique gives the distribution of the 36 possible triads for each vertex of a directed graph. Two vertices are equivalent if they have the same distribution of triads. Any technique which classifies vertices by distributions of structural properties is a relaxation of the concept of automorphic equivalence. Two automorphically equivalent vertices share exactly the same structural properties and so any techniques which concentrates on a set of properties can be viewed as a relaxation of automorphic equivalence. Since Hummell and Sodeur only consider triads, this can be equated to the local regular equivalence discussed earlier in this section. Specifically it can be viewed as a relaxation of 1-local automorphic equivalence.

Of course, since automorphically equivalent vertices share the same structural properties, all label independent positional methods can be viewed as relaxations of automorphic equivalence. Borgatti and Everett (1992a) refer to this collection of techniques as positions defined as structural isomorphism, as a contrast to those defined in terms of structural equivalence.

11. CONCLUSIONS

The mathematical theory of regular equivalence has been developed considerably since the seminal papers of Sailer (1978) and White and Reitz (1983). The development has taken place along several different fronts. One key area of progress has been in alternative characterizations of the general class of regular equivalences. This has given us a more profound understanding of the concept. Another key area has been the identification and characterization of important subsets of the class, such as the set of perfect colorations, as well as specific elements, such as automorphic and structural equivalence. It is these individual elements which are applied in substantive of social relations. Attention has also been paid to extending the notion of regular equivalence beyond simple digraphs to networks, hypergraphs, and matrices. An important area of development that we have not dealt with here is the design and implementation of computer algorithms for applying these concepts to the analysis of social network data. We take up that issue in a subsequent paper.

REFERENCES

Batagelj, V., Doreian, P., and Ferligoj, A. (1992) An optimizational approach to regular equivalence. *Social Networks* **14:** 121–135.

Borgatti, S. P. (1989) *Regular Equivalence in Graphs, Hypergraphs, and Matrices*. Ph.D. Dissertation, University of California, Irvine.

Borgatti, S. P., Boyd, J. P., and Everett, M. G. (1989) Iterated roles: Mathematics and application. *Social Networks* **11:** 159–172.

Borgatti, S. P., and Everett, M. G. (1989) The class of all regular equivalences: Algebraic structure and computation. *Social Networks* **11:** 65–88.

Borgatti, S. P., and Everett, M. G. (1992a) Notions of position in social network analysis. *Sociological Methodology*, P. Marsden (Ed.), Oxford, Basil Blackwell, 1–35.

Borgatti, S. P., and Everett, M. G. (1992b) Graph colorings and power in experimental exchange networks. *Social Networks* **14:** 287–308.

Borgatti, S. P., and Everett, M. G. (1992c) Regular blockmodels of multiway, multimode matrices. *Social Networks* **14:** 91–120.

Borgatti, S. P., and Everett, M. G. (1994) Ecological and perfect colorings. *Social Networks*, in press.

Boyd, J. P. (1991) *Social Semigroups*, Fairfax, VA, George Mason University Press.

Breiger, R. L., Boorman, S., and Arabie, P. (1975) An algorithm for clustering relational data with applications to social network analysis. *Journal of Mathematical Psychology* **12:** 329–383.

Burt, R. S. (1990) Detecting role equivalence. *Social Networks* **12:** 83–97.

Everett, M. G. (1985) Role similarity and complexity in social networks. *Social Networks* **7:** 353–359.

Everett, M. G., and Borgatti, S. P. (1991) Role colouring a graph. *Mathematical Social Science* **21:** 183–188.

Everett, M. G., and Borgatti, S. P. (1993) An extension of regular colouring of graphs to digraphs, networks and hypergraphs. *Social Networks* **15:** 237–254.

Everett, M. G., Boyd, J. P., and Borgatti, S. P. (1990) Ego-centered and local roles: A graph theoretic approach. *Journal of Mathematical Sociology* **15:** 163–172.

Hirschman, C., and Everett, M. G., The lattice of exact colorations. Unpublished manuscript.

Hummell, H., and Sodeur, W. (1987) Strukturbeschreibung von positionen in sozialen beziehungsnetzen. In *Methoden der netzwerkanalyse*, F. U. Pappi (Ed.), Munich, Oldenbourg.

Kim, K. H., and Roush, F. W. (1984) Group relationships and homomorphisms of Boolean matrix semigroups. *Journal of Mathematical Psychology* **28:** 448–452.

Lorrain, F. P., and White, H. C. (1971) Structural equivalence of individuals in networks. *Journal of Mathematical Sociology* **1:** 49–80.

Pattison, P. E. (1982) The analysis of semigroups of multirelational systems. *Journal of Mathematical Psychology* **25:** 87–118.

Sailer, L. D. (1978) Structural equivalence: Meaning and definition, computation and application. *Social Networks* **1:** 73–90.

Sachs, H. (1966) Uber Teiler, Faktoren und charakteristische Polynome von Graphen. Teill I. *Wiss. Z. TH Ilmenau* **12:** 7–12.

White, D. R., and Reitz, K. P. (1983) Graph and semigroup homomorphisms on networks of relations. *Social Networks* **5:** 193–235.

White, H. C., Boorman, S. A., and Breiger, R. L. (1976) Social structure from multiple networks: I. Blockmodels of roles and positions. *American Journal of Sociology* **81:** 730–780.

Winship, C. (1988) Thoughts about roles and relations: An old document revisited. *Social Networks* **10:** 209–231.

Winship, C., and Mandel, M. J. (1983) Roles and positions: A critique and extension of the blockmodelling approach. In *Sociological Methodology 1983–1984*, S. Leinhardt (Ed.), San Francisco, Jossey-Bass, 314–344.

Journal of Mathematical So
Reprints available directly
Photocopying permitted by

Printed in Malaysia

SEMIRINGS

Depar

J

In the paper four semirings
The closures of the matr
decide whether the graph is
The closure of relational
length and the number of
on $u-v$ geodesics. The alg
given.

KEY WORDS: Balanced s
Freeman's centrality indices

There are several app
ple, the semiring ({0,
edness matrix (Harar
$(\{0,u,m\},\max,\min)$, 0
can be used to analyse

In this paper we sh
analysis problems:

- *balance* and *cluster*
  balanced or clustera
- *geodetic* semiring to
  the number of $u-v$
- *geosetic* semiring for
  vertices u and v the

SEMIRINGS

An algebraic structure
Ullman, 1976; Burkard

*Supported in part by the M